Rigidity of Analytic Functions at the Boundary

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Abstract

A new elementary proof for a theorem of D. Burns and S. Krantz on the rigidity of the analytic self maps of the unit disc was recently discovered by L. Baracco, D. Zaitsev, and G. Zampieri. We use their argument to generalize Burns-Krantz theorems on the unit disc and on the unit ball of $\mathbb{C}^n$.

Key words: Rigidity, holomorphic functions, unit disc.

1 Introduction

A theorem of D. Burns and S. Krantz ([2, theorem 2.1]) states that if an analytic self-map f of the unit disc D satisfies

$$f(z) = z + O(|z-1|^4) \quad \text{on } D$$

then $f(z) = z$. We will refer to this theorem as Burns-Krantz theorem. The Burns-Krantz theorem was generalized to finite Blaschke products by D. Chelst [3]. The proof of both Burns-Krantz theorem and the that of the theorem of Chelst uses Hopf lemma. In a recent paper, [1], L. Baracco, D.Zaitsev and G. Zampieri gave a new and elementary proof for the Burns-Krantz theorem ([1, proposition 3.1.]). Using this we are able to prove the following more general rigidity theorem at the boundary.

Theorem 1.1 *Let φ be an analytic function on the unit disc D, having exactly n zeros, counting multiplicities, in D. Let f be an analytic function on D such that $|f| \leq |\varphi|$ on ∂D. Suppose there are n distinct points τ_j on ∂D and positive integers m_j, $j = 1, \ldots, k$ with $\sum m_j = n+1$, such that φ is away from zero around τ_j and*

$$f(z) = \varphi(z) + o(|z-\tau_j|^{2m_j-1}) \quad \text{as } D \ni z \to \tau_j \tag{1}$$

for each j. Then $f \equiv \varphi$ on D

The conditions of the theorem 1.1 already exist in the Burns-Krantz theorem and Chelst's theorem as any finite Blaschke product f have modulus 1 on ∂D and the size of the set $f^{-1}(\tau)$, $\tau \in \partial D$, is equal to the number of zeros of f in D (counting multiplicities).

Let B^n denote the unit ball of $\mathbb{C}^n$ and $B_r^n(\boldsymbol{\tau})$ the ball of radius $r > 0$ around $\boldsymbol{\tau} \in \mathbb{C}^n$. Another theorem of Burns and Krantz ([2, theorem 3.1]) states that if Φ is a holomorphic map from the unit ball B^n of $\mathbb{C}^n$ ($n \geq 2$) into itself which satisfies

$$\Phi(\boldsymbol{z}) = \boldsymbol{z} + O(|\boldsymbol{z} - \mathbf{1}|^3)$$

on B^n then $\Phi(\boldsymbol{z}) = \boldsymbol{z}$ for all $\boldsymbol{z} \in B^n$.

A similar approach lets us generalize this in the following fashion.

Theorem 1.2 *Let Φ be a holomorphic map from the unit ball B^n of $\mathbb{C}^n$ into $\mathbb{C}^m$ with polynomial components of total degree at most s and none of which vanish at $\mathbf{1}$. If F is another holomorphic map on B^n such that $|F(\boldsymbol{z})| \leq |\Phi(\boldsymbol{z})|$ for all $\boldsymbol{z} \in \partial B^n \cap B_\varepsilon^n(\mathbf{1})$ for some $\varepsilon > 0$ and*

$$F(\boldsymbol{z}) = \Phi(\boldsymbol{z}) + o(|\boldsymbol{z} - \mathbf{1}|^{2s+1}) \quad \text{as } B^n \ni \boldsymbol{z} \to \mathbf{1} \tag{2}$$

then $F \equiv \Phi$ on B^n.

Note that in this theorem we require the strong condition $|F| \leq |\Phi|$ to hold only on a neighborhood of $\mathbf{1}$ in ∂B^n.

2 Proofs

2.1 Proof of Theorem 1.1

Let α_j, $j = 1, \ldots, n$ be all the roots of φ in D repeated, if necessary, to count the multiple roots. Then we can write

$$\varphi = (z - \alpha_1) \cdots (z - \alpha_n) \cdot u(z)$$

where $u(z)$ is an analytic function on D having no zeros on D. Put $\psi = \varphi/u$ and $g = f/u$. Since $|g(z)| \leq |\psi(z)|$ and ψ is a polynomial, g is a bounded analytic function on D and since u is away from zero around τ_j,

$$\psi(z) - g(z) = o((z - \tau_j)^{2m_j - 1}) \quad \text{as } D \ni z \to \tau_j$$

for each $j = 1, \ldots, k$. Let, for $\tau \in \mathbb{C}$ and $\varepsilon > 0$, $B_\varepsilon(\tau)$ be the disc with center τ and radius ε. Set

$$B_\varepsilon = \bigcup_{j=1}^{k} B_\varepsilon(\tau_j)$$

Now, observe that

$$\mathrm{Re}\left(\overline{\psi(e^{i\theta})} \cdot \frac{\psi(e^{i\theta}) - g(e^{i\theta})}{\prod_j |e^{i\theta} - \tau_j|^{2m_j}}\right) \geq 0 \tag{3}$$

For all $\theta \in [0, 2\pi]$ except for those values of θ corresponding to τ_j. Now we will show that the integral of the function in (3) on θ from 0 to 2π is 0:

$$\begin{aligned}
&\int_{\substack{0 \leq \theta \leq 2\pi \\ e^{i\theta} \notin B_\varepsilon}} \overline{\psi(e^{i\theta})} \cdot \frac{\psi(e^{i\theta}) - g(e^{i\theta})}{\prod_j |e^{i\theta} - \tau_j|^{2m_j}} d\theta \\
&\qquad = \int_{\substack{0 \leq \theta \leq 2\pi \\ e^{i\theta} \notin B_\varepsilon}} \left(\prod_j (-e^{i\theta}\tau_j)^{m_j}\right) \overline{\psi(e^{i\theta})} \cdot \frac{\psi(e^{i\theta}) - g(e^{i\theta})}{\prod_j (e^{i\theta} - \tau_j)^{2m_j}} d\theta \\
&\qquad = -i\left(\prod_j (-\tau_j)^{m_j}\right) \int_{\partial D \setminus B_\varepsilon} z^n \overline{\psi(z)} \cdot \frac{\psi(z) - g(z)}{\prod_j (z - \tau_j)^{2m_j}} dz.
\end{aligned} \tag{4}$$

For $z \in \partial D$, $z^n \overline{\psi(z)} = (1 - \overline{\alpha}_1 z) \cdots (1 - \overline{\alpha}_n z)$ which is the boundary function of the analytic function

$$\psi_1(z) = \prod_{j=1}^{n} (1 - \overline{\alpha}_j z) \quad z \in D.$$

Now, ignoring the constant, we can rewrite integral in (4) as

$$\int_{\partial D \setminus B_\varepsilon} \psi_1(z) \cdot \frac{\psi(z) - g(z)}{\prod_j (z - \tau_j)^{2m_j}} dz = \int_{\partial B_\varepsilon \cap D} \psi_1(z) \cdot \frac{\psi(z) - g(z)}{\prod_j (z - \tau_j)^{2m_j}} dz$$

because the integrand is analytic in D. The last integral decomposes into the sum

$$\sum_{j=1}^{k} \int_{\partial B_\varepsilon(\tau_j) \cap D} \psi_1(z) \cdot \frac{\psi(z) - g(z)}{\prod_{j=1}^{k} (z - \tau_j)^{2m_j}} dz$$

and because of the condition (1) each integral in the sum goes to zero as $\varepsilon \to 0$. This means that $|\psi| = \mathrm{Re}(\overline{\psi} g)$ almost everywhere on ∂D but since $|\psi| \geq |g|$ on ∂D this is possible only when $\psi = g$ almost everywhere on ∂D. Since both ψ and g are bounded functions, $\psi \equiv g$ on D. Therefore $\varphi \equiv f$ on D.

Now we will show that the boundary condition given by the equation (1) is the best possible. To be precise, for a given bounded analytic function φ on D which is away from zero near the boundary ∂D and having zeros $\alpha_j \in D$

with multiplicities $s_j \in \mathbb{Z}_+$, $j = 1, 2, \ldots, k$, we will construct a function f analytic on D satisfying $|f(e^{i\theta})| < |\varphi(e^{i\theta})|$ for almost all $\theta \in [0, 2\pi]$ and for each $\tau \in \partial D$, if we put ν_τ for the sum of those s_j for which $\frac{1+\alpha_j}{1+\overline{\alpha}_j} = \tau$ then

$$f(z) = \varphi(z) + O(|z-\tau|^{2\nu_\tau}) \quad \text{for } z \in D \tag{5}$$

and

$$f(z) = \varphi(z) + O(|z-1|^{2\nu_1+1}) \quad \text{for } z \in D.$$

The biggest part of the problem is covered by the following proposition:

Proposition 2.1 *For $\alpha_j \in D - \{0\}$, $s \in \mathbb{Z}_{\geq 0}$ and $s_j \in \mathbb{Z}_+$, $j = 1, \ldots, k$ we have*

$$\begin{aligned}\left| z^s \prod_{j=1}^{k} (z-\alpha_j)^{s_j} + h(z)(z-1)^{2s+1} \prod_{j=1}^{k} \left(z - \frac{1+\alpha_j}{1+\overline{\alpha}_j} \right)^{2s_j} \right| \\ < \left| z^s \prod_{j=1}^{k} (z-\alpha_j)^{s_j} \right|\end{aligned} \tag{6}$$

for almost all $z \in \partial D$, where $h(z) = (-1)^n c \prod_j \frac{(1+\overline{\alpha}_j)^{2s_j}}{(1-\overline{\alpha}_j z)^{s_j}}$, $n = s + \sum_j s_s$ and $0 < c < 1/2^{-2n}$.

Proof. For $\alpha \in D$ we denote by γ_α the automorphism $z \mapsto \frac{z-\alpha}{1-\overline{\alpha}z}$. Put n for $s + \sum_j s_j$. Then for $z \in \partial D$ and $c > 0$

$$\begin{aligned}&\left| z^s \prod_j \gamma_{\alpha_j}^{s_j} + (-1)^n c (z-1)^{2s+1} \prod_j (\gamma_j - 1)^{2s_j} \right| \\ &\quad = 1 + 2(-1)^n c \,\mathrm{Re} \left((\overline{z}(z-1)^2)^s (z-1) \prod_j (\overline{\gamma}_{\alpha_j} (\gamma_{\alpha_j} - 1)^2)^{s_j} \right) \\ &\qquad + c^2 |z-1|^{4s+2} \prod_j |\gamma_{\alpha_j} - 1|^{4s_j} \\ &\quad = 1 - 2^{n+1} c (1 - \mathrm{Re} z)^{s+1} \prod_j (1 - \mathrm{Re}\gamma_{\alpha_j})^{s_j} \\ &\qquad + 2^{2n+1} c^2 (1 - \mathrm{Re} z)^{2s+1} \prod_j (1 - \mathrm{Re}\gamma_{\alpha_j})^{2s_j}\end{aligned} \tag{7}$$

$$\tag{8}$$

Observe that if $c < 2^{-2n}$, (7) is less than 1. So we have

$$\left| z^s \prod_j \gamma_{\alpha_j}^{s_j} + (-1)^n c (z-1)^{2s+1} \prod_j (\gamma_j - 1)^{2s_j} \right| \leq \left| z^s \prod_j \gamma_{\alpha_j}^{s_j} \right| \quad \forall z \in \partial D. \tag{9}$$

Multiplying both sides of (9) by $\prod_j (1 - \overline{\alpha}_j z)^{s_j}$ we obtain (6). $\square$

Now write $\varphi(z) = u(z)(z-\alpha_1)^{s_1}\cdots(z-\alpha_k)^{s_k}$ where $u(z)$ is non-vanishing analytic on D (note that, here, we allow any one α_j to be 0). It is now easy to see that the function,

$$f(z) = u(z)\left(\prod_{j=1}^{k}(z-\alpha_j)^{s_j} + h(z)\cdot(z-1)\prod_{j=1}^{k}\left(z-\frac{1+\alpha_j}{1+\overline{\alpha}_j}\right)^{2s_j}\right)$$

meets the requirements, where h is as in the proposition.

2.2 Proof of Theorem 1.2

In this section, $\langle\cdot,\cdot\rangle$ will denote the standard inner product (conjugate linear in the first variable) on $\mathbb{C}^k$ for any dimension k and $|\cdot|$ will be the corresponding norm.

In the proof, we use the argument of the proof of [1, proposition 3.1] again.

For a complex $(n-1)$-tuple $\alpha = (\alpha_2,\ldots,\alpha_n)$, set

$$Z_\alpha = \{\boldsymbol{z} = (z_1,\ldots,z_n) \in \mathbb{C}^n : z_j = \alpha_j(z_1-1),\ j = 2,3,\ldots n\}.$$

For each $\alpha \in \mathbb{C}^{n-1}$ the set $Z_\alpha \cap B^n$ is a disc which has the following parametric description,

$$Z_\alpha \cap B^n = \left\{\left(\frac{\zeta+|\alpha|^2}{1+|\alpha|^2}, \frac{\alpha_2\zeta-\alpha_2}{1+|\alpha|^2}, \cdots, \frac{\alpha_n\zeta-\alpha_n}{1+|\alpha|^2}\right) : \zeta \in D\right\}.$$

So if $|\alpha| > 2/\varepsilon = R$ then $Z_\alpha \cap B^n \subset B^n_\varepsilon(\mathbf{1})$ with $\partial(Z_\alpha\cap B^n) \subset \partial B^n \cap B^n_\varepsilon(\mathbf{1})$.

Fix an α with $|\alpha| > R$. Define functions φ and f, for $\zeta \in D$, by

$$\varphi(\zeta) = \Phi\left(\frac{\zeta+|\alpha|^2}{1+|\alpha|^2}, \frac{\alpha_1\zeta-\alpha_1}{1+|\alpha|^2}, \ldots, \frac{\alpha_n\zeta-\alpha_n}{1+|\alpha|^2}\right) \quad \text{and}$$
$$f(\zeta) = F\left(\frac{\zeta+|\alpha|^2}{1+|\alpha|^2}, \frac{\alpha_1\zeta-\alpha_1}{1+|\alpha|^2}, \ldots, \frac{\alpha_n\zeta-\alpha_n}{1+|\alpha|^2}\right).$$

Clearly, both φ and f are holomorphic maps from the unit disc into $\mathbb{C}^n$ satisfying $|f| \le |\varphi|$ on ∂D and

$$f(\zeta) = \varphi(\zeta) + o(|\zeta-1|^{(2s+1)}) \quad \text{as } D \ni \zeta \to 1.$$

So we have

$$\mathrm{Re}\left\langle \varphi(e^{i\theta}), \frac{\varphi(e^{i\theta}) - f(e^{i\theta})}{|e^{i\theta}-1|^{2s+2}}\right\rangle \ge 0 \quad \text{for all } \theta \in [0,2\pi] \tag{10}$$

We repeat the same argument as in 2.2 on

$$\int_{\substack{0 \le \theta \le 2\pi \\ e^{i\theta} \notin B_\varepsilon(1)}} \left\langle \varphi(e^{i\theta}), \frac{\varphi(e^{i\theta}) - f(e^{i\theta})}{|e^{i\theta} - 1|^{2s+2}} \right\rangle d\theta$$

to show that it is approaches to zero as $\varepsilon \to 0^+$ and conclude that $f \equiv \varphi + \eta$ for some holomorphic curve η which is normal to φ on ∂D. But this contradicts with the fact that $|f| \le |\varphi|$ on ∂D unless $\eta \equiv 0$. So we must have $f \equiv \varphi$ on D (or equivalently $F \equiv \Phi$ on $Z_\alpha \cap B^n$).

Now it is enough to show that the set

$$\bigcup_{|\alpha|>R} (Z_\alpha \cap B^n)$$

contains an open subset of B^n, but this is evident as

$$\bigcup_{|\alpha|>R} (Z_\alpha \cap B^n) = \left(\bigcup_{|\alpha|>R} Z_\alpha - \{\mathbf{1}\} \right) \cap B^n$$

which is the nonempty intersection of two open subsets of $\mathbb{C}^n$. This concludes the proof of the theorem 1.2.

The argument of L. Baracco, D. Zaitsev and G. Zampieri proves to be very fruitful from which one can deduce several types of rigidity theorems. One of these is the following, which can easily be proven using the above setting.

Proposition 2.2 *Let φ be a holomorphic function on B^n which is away from 0 near the point $\mathbf{1}$. Let f be another holomorphic function on B^n such that $|f(\boldsymbol{z})| \le |\varphi(\boldsymbol{z})|$ for $\boldsymbol{z} \in \partial B^n$ near $\mathbf{1}$ and*

$$f(\boldsymbol{z}) = \varphi(\boldsymbol{z}) + o(|\boldsymbol{z} - \mathbf{1}|) \quad \text{as } B^n \ni \boldsymbol{z} \to \mathbf{1}.$$

Then $f \equiv \varphi$.

References

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